

Two important examples of nonlinear oscillators

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We discuss a classical nonlinear oscillator, which is proved to be a superintegrable system for which the bounded motions are quasiperiodic oscillations and the unbounded (scattering) motions are represented by hyperbolic functions. This oscillator can be seen as a position-dependent mass system and we show a natural quantization prescription admitting a factorization with shape invariance for the $n = 1$ case, and then the energy spectrum is found. Other isochronous systems which can also be considered as a generalization of the harmonic oscillator and admit a nonstandard Lagrangian description are also discussed.

1 Introduction

The harmonic oscillator is a system playing a privileged rôle both in classical and quantum mechanics. It is almost ubiquitous in Physics and appears in many physical applications running from condensed matter to semiconductors (see e.g. [1] for references to such problems). The dynamical evolution of the classical system in one dimension is given by

$$\frac{dq}{dt} = v, \quad \frac{dv}{dt} = -\omega^2 q,$$

and admits a Lagrangian formulation with $L = (1/2)(v^2 - \omega^2 q^2)$, the general solution of the equations of motion being

$$q = q_0 \cos \omega t - \frac{v_0}{\omega} \sin \omega t = A \cos(\omega t + \varphi)$$

and therefore the solutions are periodic with angular frequency ω , while A and φ are arbitrary. This is the main characteristic of the classical system. As far as the quantum system is concerned, the eigenvalues of the Hamiltonian, which is given by $H = (1/2)(p^2 + \omega^2 q^2)$, are equally spaced. We should also remark that the

natural extensions to two dimensions, given by

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (\omega_1^2 x^2 + \omega_2^2 y^2)$$

admits two constants of motion in involution, $I_1 = E_x = \frac{1}{2} (p_x^2 + \omega_1^2 x^2)$, $I_2 = E_y = \frac{1}{2} (p_y^2 + \omega_2^2 y^2)$, and therefore it is completely integrable in the sense of Liouville. Moreover it has been proved that, when ω_1 and ω_2 are rationally related, i.e. $\omega_1 = n_1 \omega_0$, $\omega_2 = n_2 \omega_0$, with $n_1, n_2 \in \mathbb{N}$, there exist a new constant of motion and the system is superintegrable. Actually the complex function, $J = K_x^{n_2} (K_y^*)^{n_1}$ with $K_x = p_x + i n_1 \omega_0 x$ and $K_y = p_y + i n_2 \omega_0 y$, is a constant of the motion.

Our aim is to comment on some possible generalizations of this system from the perspective of the theory of the symmetry, i.e. trying to preserve the fundamental symmetry properties.

2 A position-dependent mass nonlinear oscillator

A often used generalization was proposed by Mathews and Lakshmanan [2, 3] as a one-dimensional analogue of some models of quantum field theory [4, 5]. It is described by a Lagrangian

$$L = \frac{1}{2} \left(\frac{1}{1 + \lambda x^2} \right) (\dot{x}^2 - \alpha^2 x^2), \quad (1)$$

which can be considered as an oscillator with a position-dependent effective mass $m = (1 + \lambda x^2)^{-1}$ (see e.g. [6, 7] and references therein). It was proved that the general solution is also $q(t) = A \cos(\omega t + \varphi)$, but now the amplitude A depends on the frequency. More explicitly $\omega^2 (1 + \lambda A^2) = \alpha^2$. Note also that this Lagrangian is of mechanical type, the kinetic term being invariant under the tangent lift of the vector field

$$X_x(\lambda) = \sqrt{1 + \lambda x^2} \frac{\partial}{\partial x}.$$

It was recently shown in [8] that there is a generalization to n dimensions preserving the symmetry characteristics. In particular the two-dimensional generalization studied in [8] was given by the Lagrangian

$$L(\lambda) = \frac{1}{2} \left(\frac{1}{1 + \lambda r^2} \right) \left[v_x^2 + v_y^2 + \lambda (xv_y - yv_x)^2 - \alpha^2 r^2 \right], \quad r^2 = x^2 + y^2, \quad (2)$$

and it was shown to be not only integrable but also superintegrable. This is the only generalization to n dimensions for which the kinetic term is a quadratic function in the velocities that is invariant under rotations and under the two vector fields generalising the symmetries of the one-dimensional model, i.e.

$$X_1(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x}, \quad X_2(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y}.$$

This is valid for any value of λ . However, when $\lambda < 0$, $\lambda = -|\lambda|$, this function has a singularity at $1 - |\lambda|r^2 = 0$ and we restrict our dynamics to the interior of the circle $x^2 + y^2 < 1/|\lambda|$.

These two vector fields close with the generator of rotations,

$$X_J = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},$$

on a Lie algebra

$$[X_1(\lambda), X_2(\lambda)] = \lambda X_J, \quad [X_1(\lambda), X_J] = X_2(\lambda), \quad [X_2(\lambda), X_J] = -X_1(\lambda).$$

which is isomorphic either to $SO(3, \mathbb{R})$, when $\lambda > 0$, or to $SO(2, 1)$, when $\lambda < 0$, or finally to the Euclidean group in two dimensions when $\lambda = 0$.

The important property shown in [8] is that this bidimensional nonlinear harmonic oscillator is completely integrable, because one can show that, if K_1 and K_2 are the functions

$$K_1 = P_1(\lambda) + i\alpha \frac{x}{\sqrt{1 + \lambda r^2}}, \quad K_2 = P_2(\lambda) + i\alpha \frac{y}{\sqrt{1 + \lambda r^2}},$$

with

$$P_1(\lambda) = \frac{v_x - \lambda J y}{\sqrt{1 + \lambda r^2}}, \quad P_2(\lambda) = \frac{v_y + \lambda J x}{\sqrt{1 + \lambda r^2}}, \quad J = x v_y - y v_x,$$

then the complex functions K_{ij} defined as $K_{ij} = K_i K_j^*$, $i, j = 1, 2$, are constants of motion. In fact the time-evolution of the functions K_1 and K_2 is

$$\frac{d}{dt} K_1 = \frac{i\alpha}{1 + \lambda r^2} K_1, \quad \frac{d}{dt} K_2 = \frac{i\alpha}{1 + \lambda r^2} K_2,$$

from which we see that the complex functions K_{ij} are constants of the motion. Therefore the system is superintegrable with the following first integrals of motion

$$I_1(\lambda) = |K_1|^2, \quad I_2(\lambda) = |K_2|^2, \quad I_3 = \Im(K_{12}) = \alpha(x v_y - y v_x).$$

The Legendre transformation for a two-dimensional Lagrangian system of mechanical type with kinetic term as in (2) is given by

$$p_x = \frac{(1 + \lambda y^2)v_x - \lambda x y v_y}{1 + \lambda r^2}, \quad p_y = \frac{(1 + \lambda x^2)v_y - \lambda x y v_x}{1 + \lambda r^2},$$

(note that $x p_y - y p_x = x v_y - y v_x$) and the general expression for the corresponding λ -dependent Hamiltonian is

$$H(\lambda) = \frac{1}{2} \left[p_x^2 + p_y^2 + \lambda (x p_x + y p_y)^2 \right] + \frac{1}{2} \alpha^2 V(x, y), \quad (3)$$

and hence the associated Hamilton–Jacobi equation is

$$\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \lambda \left(x \frac{\partial S}{\partial x} + y \frac{\partial S}{\partial y} \right)^2 + \alpha^2 V(x, y) = 2E. \quad (4)$$

This equation is not separable in (x, y) coordinates, but it was shown in [8] that there exist three particular systems of orthogonal coordinates, and three particular families of associated potentials, in which such Hamiltonians admit a Hamilton–Jacobi separability. The first system of coordinates is given by

$$(z_x, y), \quad z_x = \frac{x}{\sqrt{1 + \lambda y^2}}, \quad (5)$$

for which the Hamilton–Jacobi equation becomes:

$$(1 + \lambda z_x^2) \left(\frac{\partial S}{\partial z_x} \right)^2 + (1 + \lambda y^2)^2 \left(\frac{\partial S}{\partial y} \right)^2 + \alpha^2 (1 + \lambda y^2) V = 2(1 + \lambda y^2) E,$$

and therefore the Hamilton–Jacobi equation is separable if the potential $V(x, y)$ can be written on the form

$$V = \frac{W_1(z_x)}{1 + \lambda y^2} + W_2(y). \quad (6)$$

The potential is therefore integrable with the following two quadratic integrals of motion

$$\begin{aligned} I_1(\lambda) &= (1 + \lambda r^2) p_x^2 + \alpha^2 W_1(z_x), \\ I_2(\lambda) &= (1 + \lambda r^2) p_y^2 - \lambda J^2 + \alpha^2 \left(W_2(y) - \frac{\lambda y^2}{1 + \lambda y^2} W_1(z_x) \right). \end{aligned}$$

In a similar way, one can see, using coordinates (x, z_y) with $z_y = y(1 + \lambda x^2)^{-1/2}$, that the Hamilton–Jacobi equation is separable when the potential $V(x, y)$ is of the form

$$V = W_1(x) + \frac{W_2(z_y)}{1 + \lambda x^2}. \quad (7)$$

and the potential is integrable with the following two quadratic first integrals:

$$\begin{aligned} I_1(\lambda) &= (1 + \lambda r^2) p_x^2 - \lambda J^2 + \alpha^2 \left(W_1(x) - \frac{\lambda x^2}{1 + \lambda x^2} W_2(z_y) \right), \\ I_2(\lambda) &= (1 + \lambda r^2) p_y^2 + \alpha^2 W_2(z_y). \end{aligned}$$

Finally using polar coordinates (r, ϕ) the Hamiltonian $H(\lambda)$ is written

$$H(\lambda) = \frac{1}{2} \left[(1 + \lambda r^2) p_r^2 + \frac{p_\phi^2}{r^2} \right] + \frac{\alpha^2}{2} V(r, \phi) \quad (8)$$

and the Hamilton–Jacobi equation is given by

$$(1 + \lambda r^2) \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \phi} \right)^2 + \alpha^2 V(r, \phi) = 2E.$$

Then the equation admits separability when the potential V is of the form

$$V = F(r) + G(\phi)/r^2. \quad (9)$$

Such a potential V is integrable with the following two quadratic first integrals:

$$\begin{aligned} I_1(\lambda) &= (1 + \lambda r^2)p_r^2 + \frac{1 - r^2}{r^2}p_\phi^2 + \alpha^2 \left[F(r) + \frac{1 - r^2}{r^2} G(\phi) \right], \\ I_2(\lambda) &= p_\phi^2 + \alpha^2 G(\phi). \end{aligned}$$

Consequently, the potential

$$V = \frac{\alpha^2}{2} \left(\frac{x^2 + y^2}{1 + \lambda(x^2 + y^2)} \right)$$

is super-separable since it is separable in three different systems of coordinates (z_x, y) , (x, z_y) , and (r, ϕ) because

$$V = \frac{\alpha^2}{2} \left(\frac{1}{1 + \lambda y^2} \right) \left[\frac{z_x^2}{1 + \lambda z_x^2} + y^2 \right] = \frac{\alpha^2}{2} \left(\frac{1}{1 + \lambda x^2} \right) \left[x^2 + \frac{z_y^2}{1 + \lambda z_y^2} \right] = \frac{\alpha^2}{2} \left(\frac{r^2}{1 + \lambda r^2} \right).$$

3 The one-dimensional quantum nonlinear oscillator

We now consider the quantum case and restrict ourselves to the one-dimensional case. The first problem is to define the quantum operator describing the Hamiltonian of this position-dependent mass system, because the mass function and the momentum P do not commute and this fact gives rise to an ambiguity in the ordering of factors. It has recently been proposed to avoid the problem by modifying the Hilbert space of functions describing the system [9]. More explicitly we can consider the measure $d\mu = (1 + \lambda x^2)^{-1/2} dx$, which is invariant under the vector field $X_x(\lambda) = \sqrt{1 + \lambda x^2} \partial/\partial x$, for then the operator $P = -i\sqrt{1 + \lambda x^2} \partial/\partial x$ is selfadjoint in the space $L^2(\mathbb{R}, d\mu)$. In the case of the nonlinear oscillator in which we are interested we can consider the Hamiltonian operator

$$\hat{H}_1 = -\frac{1}{2}(1 + \lambda x^2) \frac{d^2}{dx^2} - \frac{1}{2}\lambda x \frac{d}{dx} + \frac{1}{2} \frac{\alpha^2 x^2}{1 + \lambda x^2}. \quad (10)$$

The spectral problem of such operator can be solved by means of algebraic techniques. We first remark that if β is such that $\alpha^2 = \beta(\beta + \lambda)$, then $\hat{H}'_1 = \hat{H}_1 - \beta/2$ can be factorized as a product $\hat{H}'_1 = A^\dagger(\beta) A(\beta)$ and

$$A = \frac{1}{\sqrt{2}} \left(\sqrt{1 + \lambda x^2} \frac{d}{dx} + \frac{\beta x}{\sqrt{1 + \lambda x^2}} \right), \quad (11)$$

for which its adjoint operator is

$$A^\dagger = \frac{1}{\sqrt{2}} \left(-\sqrt{1 + \lambda x^2} \frac{d}{dx} + \frac{\beta x}{\sqrt{1 + \lambda x^2}} \right). \quad (12)$$

The important point is that the partner Hamiltonian $\hat{H}'_2 = A(\beta) A^\dagger(\beta)$ is related to \hat{H}'_1 by $\hat{H}'_2(\beta) = \hat{H}'_1(\beta_1) + R(\beta_1)$ with $\beta_1 = f(\beta)$ and where f and R are the functions $f(\beta) = \beta - \lambda$ and $R(\beta) = \beta + (1/2)$. Hamiltonians admitting such factorization [10] and related to its partner in such a way are said to be shape invariant and their spectra and the corresponding eigenvectors can be found by using the method proposed by Gendenshtein [11, 12] (see also [13] for a modern presentation based on the Riccati equation). Therefore, as the quantum nonlinear oscillator is shape invariant, we can develop the method proposed in [11, 12] for finding both the spectrum and the corresponding eigenvectors. The spectrum is given by [9]:

$$E_n = n\beta - \frac{n^2}{2}\lambda + \frac{1}{2}\beta.$$

The existence of a finite or infinite number of bound states depends up on the sign of λ as also discussed in [9].

4 Periodic motions and another nonlinear oscillator

Another possible generalization of the harmonic oscillator would be to look for alternative isochronous systems. For instance one can consider a potential

$$U(x) = \begin{cases} U_1(x) & \text{if } x < 0 \\ U_2(x) & \text{if } x > 0, \end{cases}$$

where $U_2(x)$ is an increasing function and $U_1(x)$ is a decreasing function, and try to determine the explicit functions U_1 and U_2 in order to have an isochronous system. The problem of the determination of the potential when the period is known as a function of the energy was solved by Abel [14]. When the potential is symmetric the solution is unique. Therefore the only symmetric potential giving rise to isochronous motions around the origin is the harmonic oscillator. The isotonic oscillator is also symmetric and isochronous, but the origin is a singular point and not a minimum of the potential. Other nonsymmetric potentials can be used, for instance a potential given by

$$U_1(x) = \omega_1^2 x^2, \quad U_2(x) = \omega_2^2 x^2.$$

If we want to find more general solutions for the symmetric case we may consider Lagrangians of a nonstandard mechanical type, in which there is no potential term. These more general Lagrangians can also be relevant in other problems. For instance another interesting oscillator-like system has recently been studied by Chandrasekar *et al* [15]. As mentioned in that paper the oscillator-like system admits a Lagrangian formulation. We recall that there are systems admitting a Lagrangian formulation of a nonmechanical type. As an example we can consider $Q = \mathbb{R}$ as the configuration space and the Lagrangian function [16]

$$L(x, v) = (\alpha(x)v + U(x))^{-1}, \quad (13)$$

which is singular in the zero level set of the function $\varphi(x, v) = \alpha(x) v + U(x)$.

Then the Euler–Lagrange equation is

$$\alpha'(x) v + U'(x) - \alpha'(x) v = -\frac{2\alpha(x)[\alpha'(x)v^2 + U'(x)v + \alpha(x)a]}{\alpha(x)v + U(x)},$$

where v and a denote the velocity and the acceleration, respectively.

This is a conservative system the equation of motion of which can be rewritten as

$$[\alpha(x)]^2 \ddot{x} + \alpha(x) \alpha'(x) \dot{x}^2 + \frac{3}{2} \alpha(x) U'(x) \dot{x} + \frac{1}{2} U(x) U'(x) = 0.$$

The energy is given by $E_L(x, v) = -[2\alpha(x)v + U(x)][\alpha(x)v + U(x)]^{-2}$. In particular, when $\alpha(x) = 1$, the Lagrangian is $L(x, v) = [v + U(x)]^{-1}$ and the Euler–Lagrange equation reduces to

$$\ddot{x} + \frac{3}{2} U'(x) \dot{x} + \frac{1}{2} U(x) U'(x) = 0 \quad (14)$$

and the energy function turns out to be $E_L(x, v) = -[2v + U(x)][v + U(x)]^{-2}$. When $U(x) = kx^2$, the equation is

$$\ddot{x} + 3kx\dot{x} + k^2x^3 = 0,$$

and the energy is $E_L = -[2v + kx^2][v + kx^2]^{-1}$. It can be seen from the energy conservation law that the general solution is

$$x = \frac{2t}{k t^2 - E}.$$

The two-dimensional system described by $L(x, y, v_x, v_y) = [v_x + k_1 x^2]^{-1} + [v_y + k_2 y^2]^{-1}$ is superintegrable. Actually not only the energies of each degree of freedom are conserved but also the functions [16]

$$I_3 = \frac{x}{v_x + k_1 x^2} - \frac{y}{v_y + k_2 y^2}, \quad I_4 = \frac{k_2}{v_x + k_1 x^2} + \frac{k_1}{v_y + k_2 y^2} - \frac{k_1 k_2 x y}{(v_x + k_1 x^2)(v_y + k_2 y^2)}.$$

Another example is that of a nonlinear oscillator for which we were looking. The following Lagrangian depending on the parameter ω

$$L(x, v; \omega) = \frac{1}{k v_x + k^2 x^2 + \omega^2}, \quad (15)$$

produces the nonlinear Euler–Lagrange equation

$$\ddot{x} + 3kx\dot{x} + k^2x^2 + \omega^2x = 0,$$

which is the nonlinear oscillator system recently studied by Chandrasekar *et al* [15], and the energy is $E_L = -[2kv_x + k^2x^2 + \omega^2][(kv_x + k^2x^2 + \omega^2)]^{-2}$. The general solution for the dynamics, which can be found from the energy conservation, is

$$x = \frac{\omega \sqrt{E} \sin(\omega t + \phi)}{1 - k \sqrt{E} \cos(\omega t + \phi)}.$$

We have recently been able to prove [16] that in the rational case of the two-dimensional problem, for which $\omega_1 = n_1 \omega_0$ and $\omega_2 = n_2 \omega_0$, the system is superintegrable as it was the case for the harmonic oscillator. To introduce the additional constants of motion we define

$$\mathbb{K}_1 = \frac{v_x + k_1 x^2 + i n_1 \omega_0 x}{k_1 v_x + k_1^2 x^2 + n_1^2 \omega_0^2}, \quad \mathbb{K}_2 = \frac{v_y + k_2 y^2 + i n_2 \omega_0 y}{k_2 v_y + k_2^2 y^2 + n_2^2 \omega_0^2}, \quad (16)$$

and then the complex function $\mathbb{K}_1^{n_2} (\mathbb{K}_2^*)^{n_1}$ is a constant of the motion.

In summary, not only position-dependent mass generalizations of the harmonic oscillator can be interesting but there exist also systems described by Lagrangians of non-mechanical type which preserve the property of superintegrability for the harmonic oscillator with rationally related frequencies. This example points out the importance of the study of such non-standard Lagrangians.

- [1] Jiang Yu and Shi-Hai Dong, Exactly solvable potentials for the Schrödinger equation with spatially dependent mass. *Phys. Lett. A* **325** (2004), 194–198.
- [2] P.M. Mathews and M. Lakshmanan, On a unique nonlinear oscillator. *Quart. Appl. Math.* **32** (1974), 215–218.
- [3] M. Lakshmanan and S. Rajasekar, *Nonlinear Dynamics. Integrability, Chaos and Patterns*. Advanced Texts in Physics, Springer-Verlag, Berlin 2003.
- [4] R. Delbourgo, A. Salam and J. Strathdee, Infinities of nonlinear and Lagrangian theories. *Phys. Rev.* **187** (1969), 1999–2007.
- [5] K. Nishijima and T. Watanabe, Green's functions in nonlinear field theories. *Prog. Theor. Phys.* **47** (1972), 996–1003.
- [6] J.M. Lévy-Leblond, Position-dependent effective mass and Galilean invariance. *Phys. Rev A* **52** (1995), 1485–1489.
- [7] R. Koç and M. Koca, A systematic study on the exact solution of the position-dependent mass Schrödinger equation. *J. Phys. A* **36** (2003), 8105–12.
- [8] J.F. Cariñena, M.F. Rañada, M. Santander and M. Senthilvelan, A nonlinear oscillator with quasi-harmonic behaviour: two- and n -dimensional oscillators. *Nonlinearity* **17** (2004), 1941–63.
- [9] J.F. Cariñena, M.F. Rañada and M. Santander, One-dimensional model of a quantum nonlinear Harmonic oscillator. *Rep. Math. Phys.* **54** (2004), 285–93.
- [10] L. Infeld and T.E. Hull, The factorization method. *Rev. Mod. Phys.* **23** (1951), 21–68.
- [11] L.É. Gendenshtein, Derivation of exact spectra of the Schrödinger equation by means of supersymmetry. *JETP Lett.* **38**, (1983) 356–59.
- [12] L.É. Gendenshtein and I.V. Krive, Supersymmetry in quantum mechanics. *Soviet Phys. Usp.* **28** (1985), 645–66.
- [13] J.F. Cariñena and A. Ramos, Riccati equation, factorization method and shape invariance. *Rev. Math. Phys.* **12** (2000), 1279–304.
- [14] N.H. Abel, Auflösung einer mechanischen Aufgabe. *J. Reine Angew. Math* **1**, (1826) 153–57.
- [15] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, An unusual Liénard type oscillator with properties of a linear harmonic oscillator, arXiv: nlin.SI/0408054 (2004).
- [16] J.F. Cariñena, M.F. Rañada and M. Santander, Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability (Preprint, University of Zaragoza, 2005).